
Student Number

Barker College

2020 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Staff Involved:

- ALY*
- JZT*
- ARM
- AXD
- ESP
- WMD

FRIDAY 21st AUGUST

100 copies

General Instructions:

- Reading time – 10 minutes
- Working time – 120 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and / or calculations

Total marks: 70

Section I – 10 marks (pages 2 - 5)

- Attempt Multiple Choice Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 - 11)

- Attempt Questions 11 - 14
- Show all necessary working
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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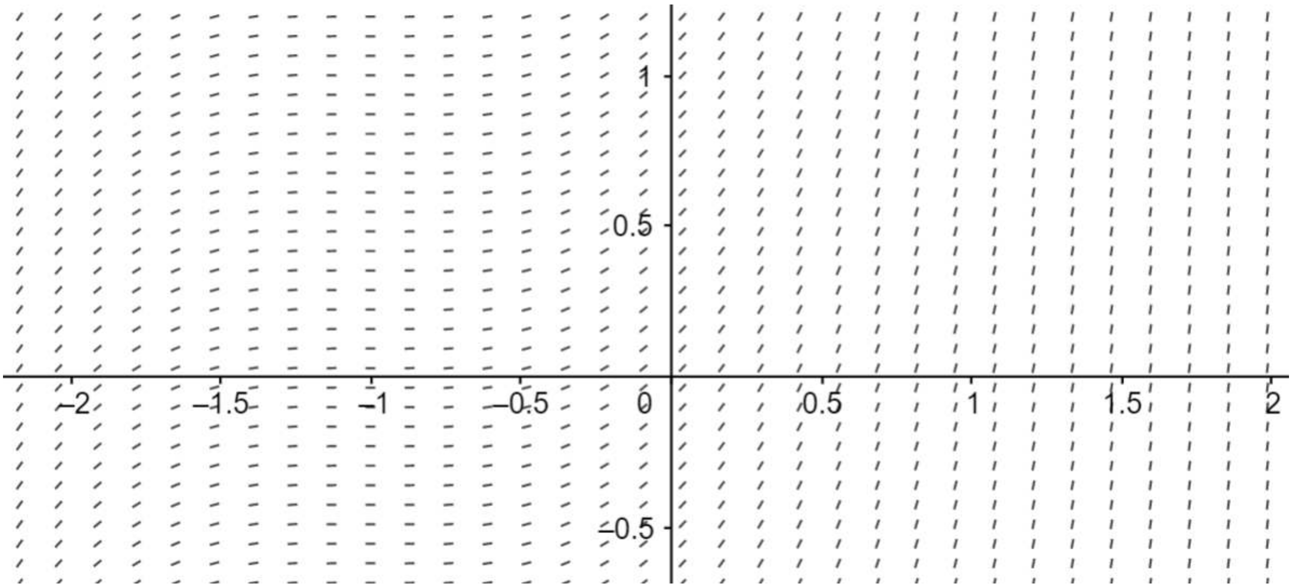
1. Which differential equation would generate the direction field shown below?



(A) $y = (x-1)^2$

(B) $y = (x+1)^2$

(C) $\frac{dy}{dx} = (x-1)^2$

(D) $\frac{dy}{dx} = (x+1)^2$

2. Find the derivative of $3\sin^{-1}(2x)$

(A) $\frac{3}{2\sqrt{1-4x^2}}$

(B) $\frac{6}{\sqrt{1-4x^2}}$

(C) $\frac{3}{\sqrt{1-4x^2}}$

(D) $\frac{12}{\sqrt{1-4x^2}}$

3. The polynomial $P(x) = x^3 - 9x^2 - 21x + 245$ has a double root. The double root is:

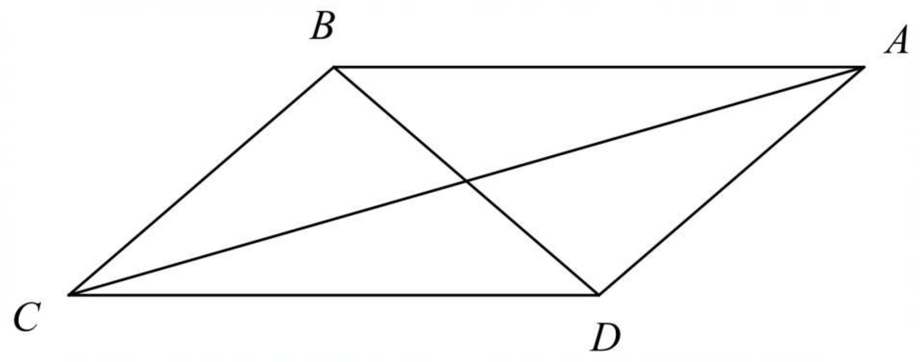
(A) -7

(B) -5

(C) -1

(D) 7

4. If *ABCD* is a parallelogram, which vector is equal to $\overrightarrow{AB} + \overrightarrow{BD}$?



(A) $\overrightarrow{DA}$

(B) $\overrightarrow{BC}$

(C) $\overrightarrow{AC}$

(D) $\overrightarrow{CD}$

5. In how many ways can 7 people be chosen from a group of 16 people and then arranged in a circle?

(A) $\frac{15!}{9!}$

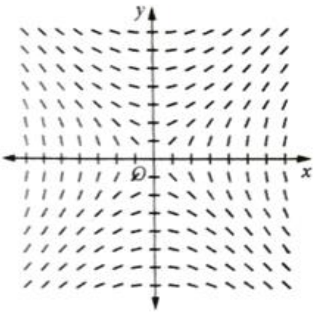
(B) $\frac{15!}{9!7!}$

(C) $\frac{16!}{9!}$

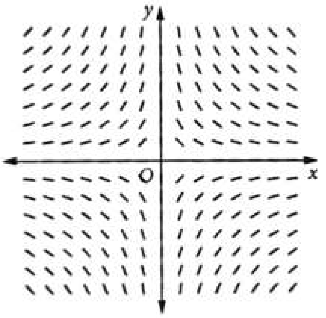
(D) $\frac{16!}{9!7}$

6. The slope field of $xy' - y = 0$ could be

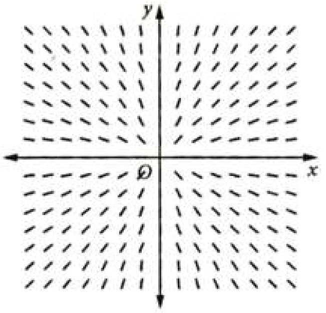
(A)



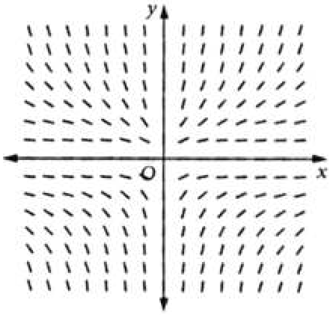
(B)



(C)



(D)



7. When the polynomial $P(x)$ is divided by $(x+2)(x-5)$ the remainder is $3x-6$. What is the remainder when $P(x)$ is divided by $(x+2)$?

(A) -2

(B) -7

(C) -12

(D) -28

8. Using the substitution $u = x^2 + 1$ or otherwise, evaluate $\displaystyle\int_0^1 \frac{x}{(x^2+1)^2}\,dx$.

(A) $-\frac{1}{4}$

(B) $-\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{1}{2}$

9. Find the coefficient of x^5 in the binomial expansion of

$$\left(\frac{1}{x^3} + 2x^2\right)^{15}$$

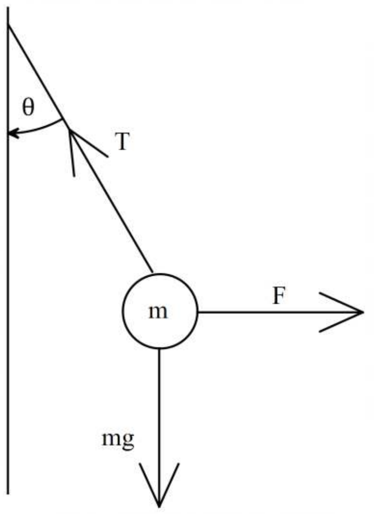
(A) 96 096

(B) 1 647 360

(C) 2 562 560

(D) 3 075 072

10. A mass of 4kg is attached to a wall by a light, inextensible string that makes an angle of θ with the wall. A force, F, is applied to the mass away from the wall at an angle perpendicular to the wall. If the tension, T, in the string is 80N, what is the force F that is applied if the mass is in equilibrium? Assume acceleration due to gravity is 10ms^{-2}.



(A) $40\sqrt{3}$N

(B) 40N

(C) $40\sqrt{2}$N

(D) 20N

End of Section I

Section II

60 marks
Attempt Questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
Answer Question 11 (e) and (f) (ii), (iii) on the separate answer page provided.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[START A NEW BOOKLET]** **Marks**

(a) Solve for x **3**

$$\frac{1}{x-2} > -1$$

(b) Find the general solution of the differential equation $y' = \frac{-2x}{y}$. Answer with y as the subject. **2**

(c) A particle moves so that its displacement x from the origin is a function of time t. **2**

Its velocity is given by the differential equation $\frac{dx}{dt} = \sin^2 3t$.

Initially, the particle is $\frac{\pi}{4}$ metres to the right of the origin.

Find the particular solution to the differential equation that gives displacement as a function of time.

(d) Evaluate **2**

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{1+4x^2}$$

Answer Question 11 (e) on the separate answer page supplied.
Hand it in with your booklet for Question 11.

(e) Sketch the slopefield for **2**

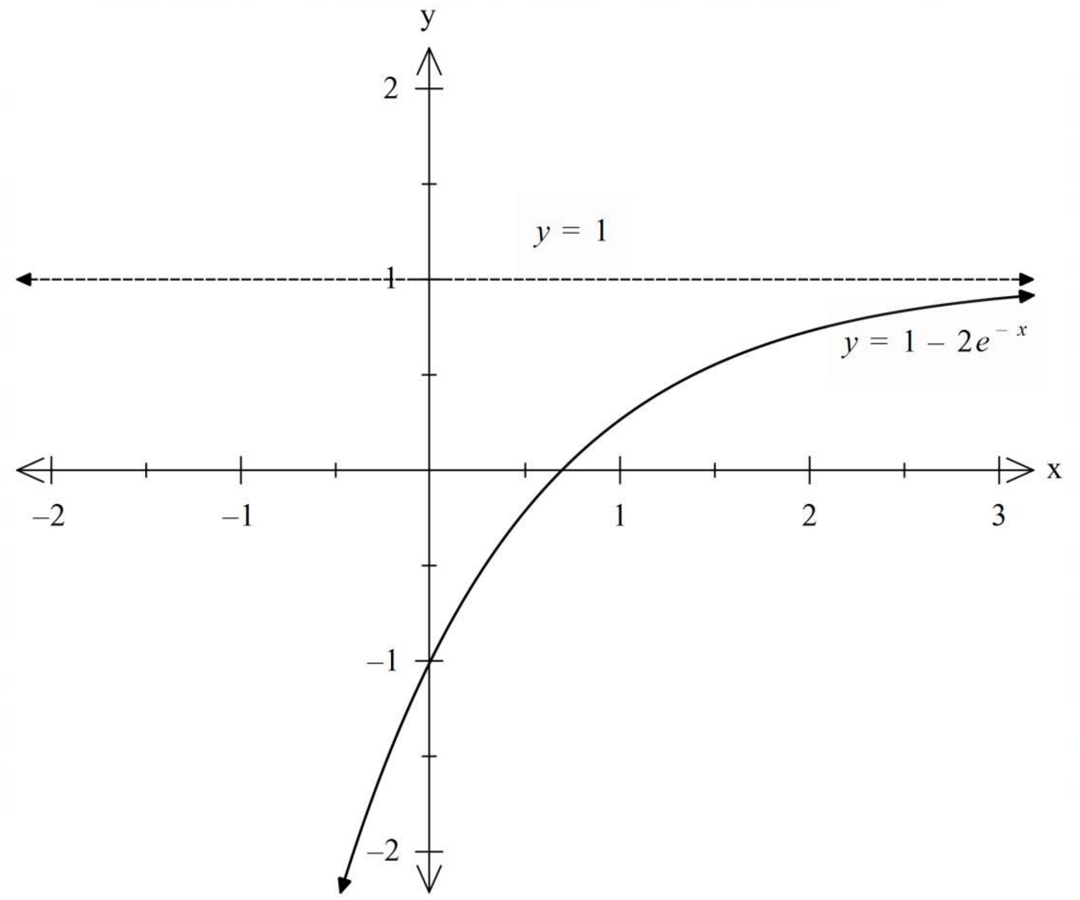
$$y' = x - y.$$

Draw one slope element at each of the points marked with a ‘ + ’.
A table of slopes is also provided on the answer page if you wish to use it.

Question 11 continues on the next page.

Question 11 continued.

(f) Consider the graph of $f(x) = 1 - 2e^{-x}$ shown below.



(i) Find the x intercept of this graph in exact form. **1**

Answer Question 11 (f) (ii) and (iii) on the separate answer page supplied. Hand it in with your booklet for Question 11.

(ii) Sketch $y = f(|x|)$. Show/label all important features. **1**

(iii) Sketch $y = \frac{1}{f(x)}$. Show/label all important features. **2**

End of Question 11

Question 12 (15 Marks) **[START A NEW BOOKLET]** **Marks**

(a) Evaluate x if the vectors $\begin{pmatrix} x \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 8 \end{pmatrix}$ are:

(i) perpendicular **1**

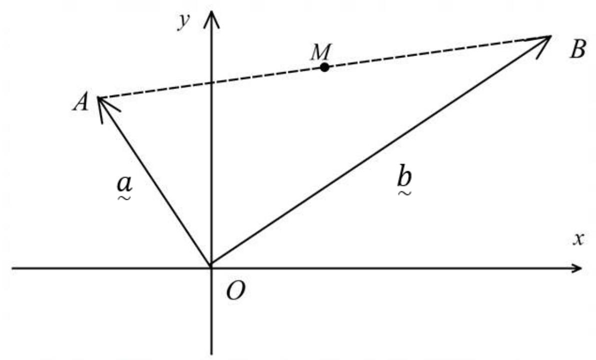
(ii) parallel **1**

(b) The mass M of a particular southern right whale is modelled as $\frac{dM}{dt} = k(55 - M)$, where M is measured in tonnes, t is the age of the whale in years, and k is a positive constant.

(i) Solve the differential equation by integration to show that the general solution is $M = 55 - Ae^{-kt}$. **2**

(ii) This whale was 1 tonne at birth, and 10 tonnes when it was 1 year old. Find the whale's mass when it is 10 years old, to the nearest tonne. **3**

(c) Given $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0,$ show, with the aid of the diagram, that $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \left|\underset{\sim}{a} - \underset{\sim}{b}\right|$, and hence show that the midpoint, M, of the hypotenuse of a right-angled triangle is equidistant from all three of its vertices. **3**



(d) (i) If $t = \tan A,$ find an expression in terms of t for $\cos 2A$. **1**

(ii) Use t formulas with the substitution $t = \tan x$ to prove that **2**

$$\sec 2x + \tan 2x = \tan(x + 45°)$$

(iii) Hence or otherwise solve $\sec 2x + \tan 2x = \frac{4}{5}$ for $0° \leq x \leq 360°$ **2**

Answer to the nearest degree.

End of Question 12

Question 13 (15 marks) **[START A NEW BOOKLET]** **Marks**

(a) Consider the function $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

(i) State the domain and range of $f(x)$. **1**

(ii) Find and sketch the inverse function $f^{-1}(x)$. **3**

(b) What is the least number of distinct integers that can be chosen from the sequence $1, 4, 7, 10, 13, 16, 19, \dots, 97, 100$ so that it is guaranteed that two of them will have a sum of 104? **2**

(c) Prove by induction that **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}, \text{ for } n \geq 1$$

(d) A ball rolling along a horizontal plane has position and velocity vectors given below:

$$\text{Position in metres: } \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j},\ t \geq 0, \qquad 0 < y < 1$$

$$\text{Velocity in } ms^{-1}\text{: } \dot{\underset{\sim}{r}} = 2y\sqrt{y}\underset{\sim}{i} + y(1-y)\underset{\sim}{j}$$

The component of the velocity in the $\underset{\sim}{j}$ direction is the differential equation

$$\frac{dy}{dt} = y(1-y).$$

It can be shown that $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$. Do not prove this.

(i) Solve the differential equation $\frac{dy}{dt} = y(1-y)$ to find the general solution, and hence show that if the particle is initially at the point $(0, 0.5)$, the component of the displacement in the $\underset{\sim}{j}$ direction is: **3**

$$y = \frac{e^t}{1+e^t}$$

(ii) Show that the magnitude of the velocity vector is $y(1+y)$ and hence calculate the magnitude of the velocity at time $t = \ln 9$ seconds. **3**

End of Question 13

Question 14 (15 marks) **[START A NEW BOOKLET]** **Marks**

(a) Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

(i) Show that $\hat{\underset{\sim}{b}} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$ **1**

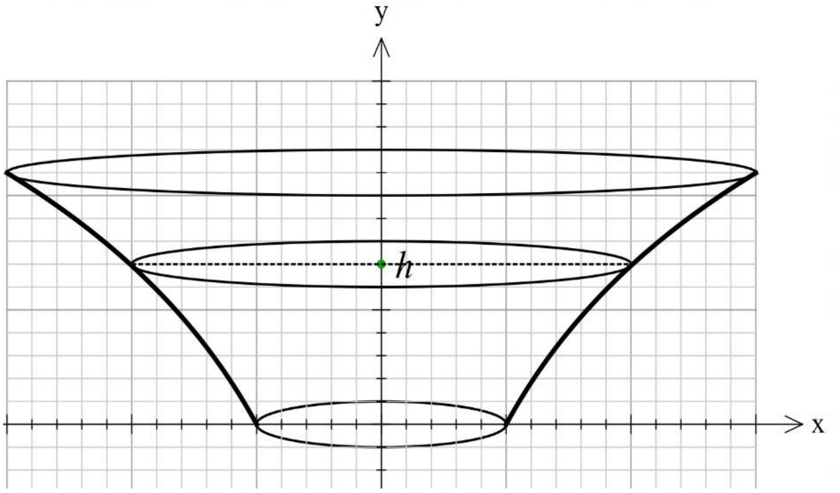
(ii) Hence, or otherwise, find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$ **2**

(iii) Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. **1**

(b) A bowl is formed by rotating the curve $y = 2\ln x$ around the y axis.

The volume of water in the bowl when the height of the water is h, can be modelled by rotating the area enclosed by the curve, the lines $y = 0$ and $y = h$, about the y axis.

The lengths x, y and h are in centimetres.



(i) Show that the volume of water in the bowl is $V = \pi(e^h - 1)$ cm^3. **2**

(ii) A tap is dripping into this bowl at a rate of 40 cm^3/minute. **4**

Find the rate at which the height of the water is increasing after 3 minutes, if the bowl is initially empty.

Give your answer in exact form.

Question 14 continues on the next page.

Question 14 continued.

(c) (i) A ball is launched from ground level into the air with an initial speed of 25 ms^{-1} and just clears a 3 metre fence. The fence is 15 metres to the right of the launch site. **3**

You may assume the equations of motion for the ball:

$$x = 25t\cos\theta\,, \qquad y = -4.9t^2 + 25t\sin\theta$$

By first finding the cartesian equation of motion, find the two possible angles of projection of the ball. Answer to the nearest **second**.

(ii) For each launch angle, calculate the speed of the ball when it reaches the fence. **2**

Which launch angle gives the ball a greater speed when it reaches the fence?

End of Paper

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Hand in with Question 11 writing booklet.

Student Number

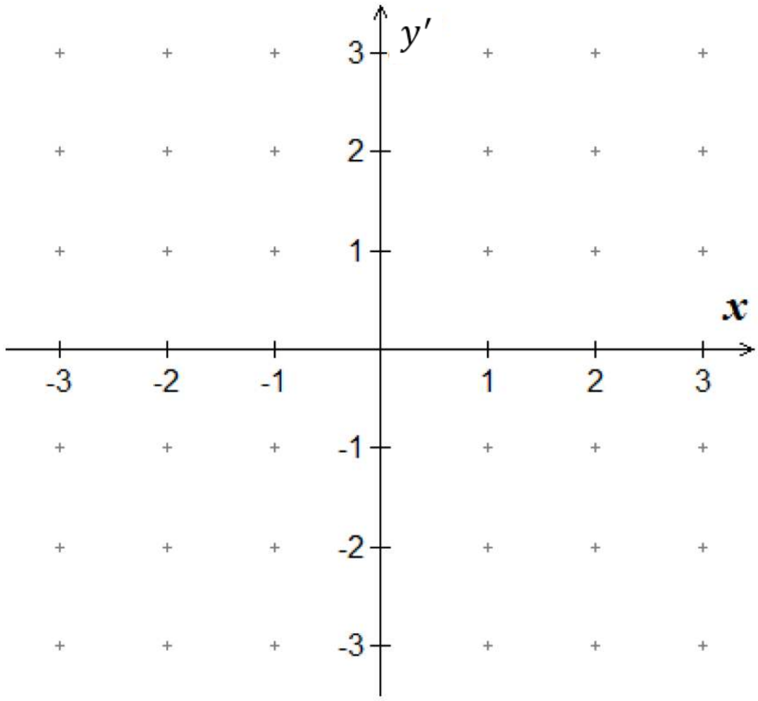
Additional Answer page for Question 11(e). Answer Question 11 (e) here.

11. (e) **On this answer page,** sketch the slopefield for $y' = x - y$. **2**

Draw one slope element at each of the points marked with a ‘ + ’.

A table of slopes is also provided if you wish to use it.

Slope field:



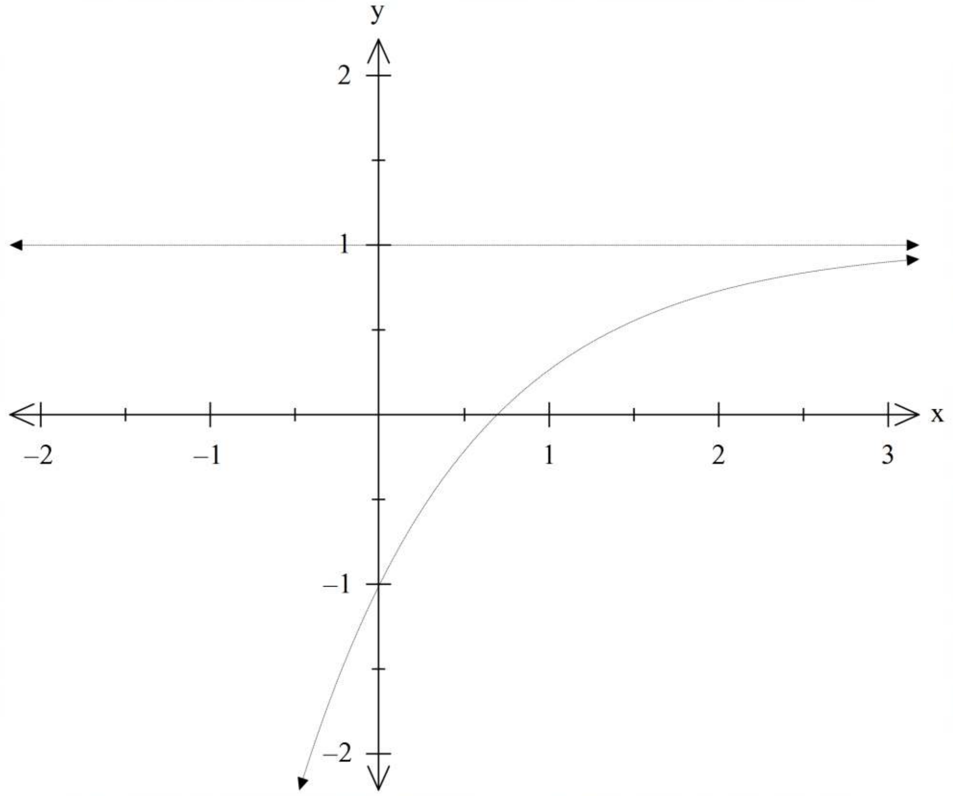
Optional Table of Slopes:

	3							
	2							
	1							
y	0							
	−1							
	−2							
	−3							
		−3	−2	−1	0	1	2	3
					x			

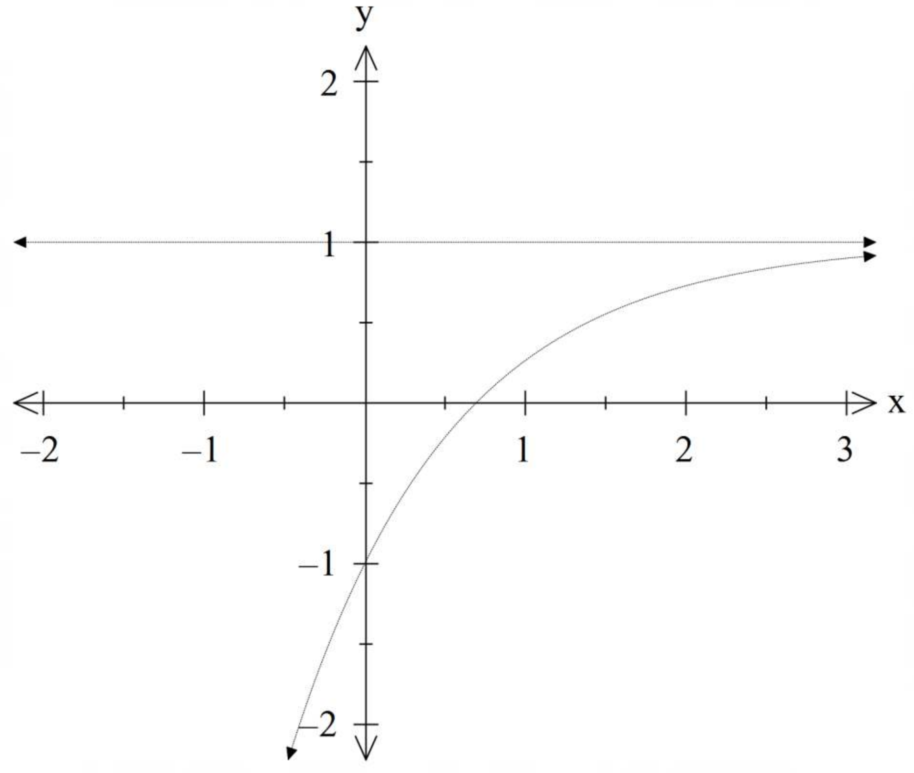
Additional Answer page for Question 11 (f) (ii), (iii).

Answer Question 11(f) (ii) and (iii) here.

11(f) (ii) **Sketch on this answer page** $y = f(|x|)$. Show/label all important features. **1**



(iii) **Sketch on this answer page** $y = \frac{1}{f(x)}$. Show/label all important features. **2**



1\. D

2\. B $\frac{d}{dx}\left(3\sin^{-1}2x\right)$

$= 3 \times 2 \times \frac{1}{\sqrt{1-(2x)^2}}$

$= \frac{6}{\sqrt{1-4x^2}}$

3\. D $P'(x) = 3x^2 - 18x - 21$

$= 3(x+1)(x-7)$

$= 0$ when $x = -1$ or 7

$P(7) = 0$ and $P'(7) = 0$

$\therefore$ Double root is 7

4\. B $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\overrightarrow{AD} = \overrightarrow{BC}$ (opp sides, parallelogram)

5\. D ${}^{16}C_7 \times 6! = \frac{16!}{7!9!} \times 6!$

$= \frac{16}{9!7}$

6\. C

7\. C $P(-2) = 0 \times -7 + 3(-2) - 6$

$= -12$

8\. C $u = x^2 + 1$ $\quad x=1\ u=2$, $x=0\ u=1$

$\underset{2}{du} = 2x\,dx$

$= \frac{1}{2}\int_1^2 \frac{1}{u^2}$

$= -\frac{1}{2}\left[\frac{1}{u}\right]_1^2 = \frac{-1}{2}\left(\frac{1}{2} - 1\right)$

$= \frac{1}{4}$

9\. D $\binom{15}{r}(x^{-3})^{15-r}(2x^2)^r$

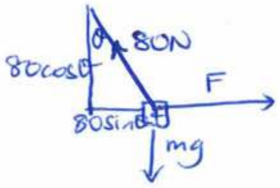
$= \binom{15}{r} x^{-45+3r+2r} 2^r$

$-45 + 5r = 5$

$r = 10$

coeff $= \binom{15}{10} 2^{10} = 3075072$

10\. A



$mg = 40$

$80\cos\theta = 40$

$\therefore \cos\theta = \frac{1}{2}$ $\theta = 60°$

$F_{horiz} \rightarrow 80\sin 60° = F$

$F = 80 \times \frac{\sqrt{3}}{2}$

$= 40\sqrt{3}$ N.

Question 11

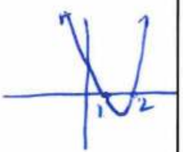
a) $\frac{1}{x-2} > -1$

$(x-2) > -(x-2)^2$

$(x-2)^2 + (x-2) > 0$

$(x-2)(x-1) > 0$

$x < 1$, $x > 2$



b) $y' = \frac{-2x}{y}$

$\int y\,dy = -2\int x\,dx$

$\frac{y^2}{2} = -\frac{2x^2}{2} + C$

$y^2 = -2x^2 + C_1$

$y = \pm\sqrt{-2x^2 + C_1}$

c) $\frac{dx}{dt} = \sin^2 3t$

$\int dx = \int \sin^2 3t\,dt$

$x = \frac{1}{2}\int(1 - \cos 6t)\,dt$

$= \frac{1}{2}\left[t - \frac{1}{6}\sin 6t\right] + C$

at $t = 0$, $x = \frac{\pi}{4}$

$\frac{\pi}{4} = \frac{1}{2}(0-0) + C \therefore C = \frac{\pi}{4}$

$\therefore x = \frac{1}{2}\left(t - \frac{1}{6}\sin 6t + \frac{\pi}{2}\right)$

d) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{1+4x^2} = \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{2\,dx}{1+4x^2}$

$= \frac{1}{2}\left[\tan^{-1}2x\right]_0^{\frac{\sqrt{3}}{2}}$

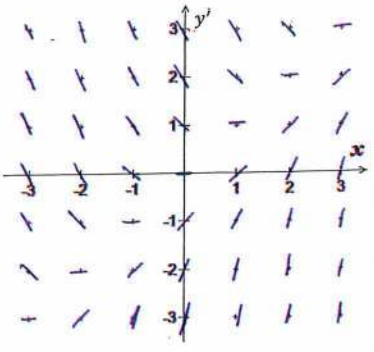
$= \frac{1}{2}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$

$= \frac{\pi}{6}$

ile

11. (e) **On this answer page,** sketch the slopefield for
$y' = x - y$.
Draw one slope element at each of the points marked with a ' + '.
A table of slopes is also provided if you wish to use it.

Slope field:



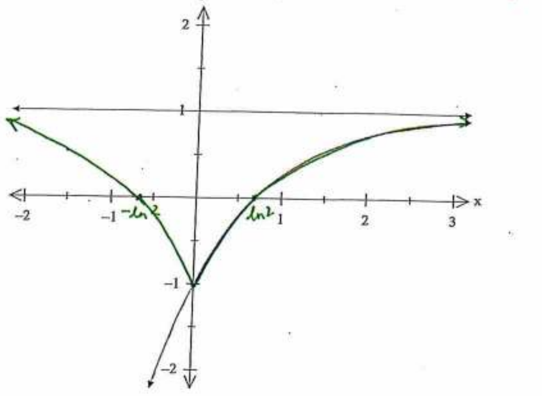
Optional Table of Slopes:

3	-6	-5	-4	-3	-2	-1	0
2	-5	-4	-3	-2	-1	0	1
1	-4	-3	-2	-1	0	1	2
y 0	-3	-2	-1	0	1	2	3
-1	-2	-1	0	1	2	3	4
-2	-1	0	1	2	3	4	5
-3	0	1	2	3	4	5	6
	−3	−2	−1	0	1	2	3

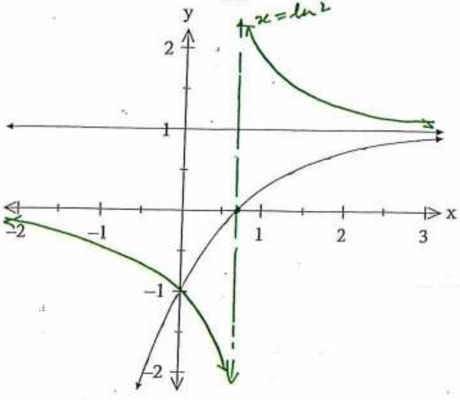
x

11f)

(ii) **Sketch on this answer page** $y = f(|x|)$. Show/label all important features. 1



(iii) **Sketch on this answer page** $y = \frac{1}{f(x)}$. Show/label all important features. 2



f) i) For x-int
$1 - 2e^{-x} = 0$
$e^{-x} = \frac{1}{2}$
$e^{x} = 2$ $x = \ln 2.$

Question 12

12a (i) perp if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\begin{pmatrix} x \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

$-6x + 24 = 0$

$x = 4$

ii) parallel if $\underset{\sim}{a} = k\underset{\sim}{b}$

$\begin{pmatrix} x \\ 3 \end{pmatrix} = k\begin{pmatrix} -6 \\ 8 \end{pmatrix}$

$x = -6k$

and $3 = 8k$

$\therefore k = \frac{3}{8}$

$x = -6 \times \frac{3}{8} = \frac{-9}{4}.$

12b (i) $\int \frac{dM}{55-M} = \int k\,dt$

$-\ln|55-M| = kt + c$

$\ln|55-M) = -kt + c_1$

$e^{-kt+c_1} = |55-M|$

$|55-M| = e^{-kt}e^{c_1}$

Let $A = \pm e^{c_1}$ $55 - M = Ae^{-kt}$

$M = 55 - Ae^{-kt}$ (shown)

(ii) $M = 55 - Ae^{-kt}$ at $t = 0, M = 1$

$1 = 55 - A$

$\therefore A = 54$

$\therefore M = 55 - 54e^{-kt}$

at $t = 1$ $M = 10$

$10 = 55 - 54e^{-k}$

$-45 = -54e^{-k}$

$e^{-k} = \frac{45}{54}$

$e^{k} = \frac{54}{45} \rightarrow k = \ln\left(\frac{54}{45}\right)$

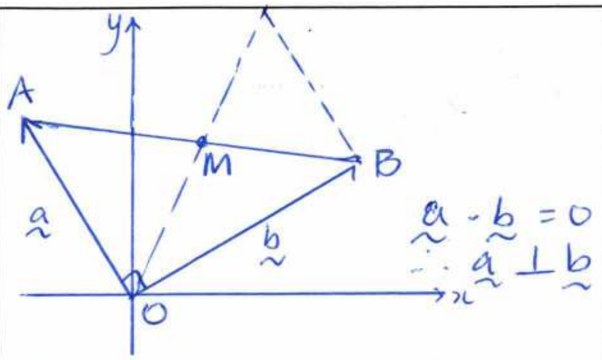
when $t = 10$

$M = 55 - 54e^{-10\ln\left(\frac{54}{45}\right)}$

$= 46.278...$

$= 46$ tonnes (nearest tonne)

12c



$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\therefore \underset{\sim}{a} \perp \underset{\sim}{b}$

$\overrightarrow{BA} = -\underset{\sim}{b} + \underset{\sim}{a} = \underset{\sim}{a} - \underset{\sim}{b}$

By Pythagoras

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |-\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2$$
$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

Construct $\overrightarrow{BC}$ such that

$\overrightarrow{BC} = \overrightarrow{OA} = \underset{\sim}{a}$ (same mag. and direction)

$\therefore \overrightarrow{OC} = \underset{\sim}{b} + \underset{\sim}{a}$

by Pythagoras

$$|\overrightarrow{OC}|^2 = |\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$
$$= |\underset{\sim}{a} - \underset{\sim}{b}|^2 \text{ (from above)}$$

$\therefore |\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a} - \underset{\sim}{b}|^2$

$\overrightarrow{BM} = \overrightarrow{MA} = \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$

(M is midpoint of $\overrightarrow{BA}$)

$|\overrightarrow{BM}| = |\overrightarrow{MA}| = \frac{1}{2}|\underset{\sim}{a} - \underset{\sim}{b}|$

$$\overrightarrow{OM} = \underset{\sim}{a} + \frac{1}{2}\overrightarrow{AM} = \underset{\sim}{a} - \frac{1}{2}\overrightarrow{MA}$$
$$= \underset{\sim}{a} - \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$$
$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$$

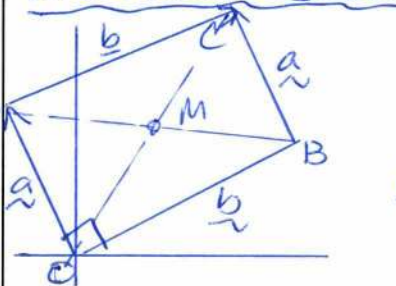
$\therefore |\overrightarrow{OM}| = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

$\because |\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a} - \underset{\sim}{b}|^2$

$\therefore |\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$

$\therefore |\overrightarrow{OM}| = |\overrightarrow{BM}| = |\overrightarrow{MA}|$

12c alternate solution.



$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\therefore \underset{\sim}{a} \perp \underset{\sim}{b}$

Construct $\overrightarrow{AC}$ parallel and equal in mag. to $\overrightarrow{OB}$

$\therefore \overrightarrow{AC} = \overrightarrow{OB} = \underset{\sim}{b}$

Similarly construct $\overrightarrow{BC}$ such that $\overrightarrow{BC} = \overrightarrow{OA} = \underset{\sim}{a}$

$\therefore$ OBCA is a rectangle (opposite sides equal and parallel and 1 angle a right angle).

Since AB and OC are the diagonals of a rectangle, they bisect each other and are equal.

$\therefore |\overrightarrow{AM}| = |\overrightarrow{BM}| = |\overrightarrow{OM}|$

12d (i) $\cos 2A = \frac{1-t^2}{1+t^2}$

(ii) RTP $\sec 2x + \tan 2x = \tan(x+45°)$

$$LHS = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}$$
$$= \frac{t^2+2t+1}{1-t^2}$$
$$= \frac{(t+1)^2}{(1-t)(1+t)}$$
$$= \frac{t+1}{1-t}$$

→

12d ii) con't'd

$$RHS = \frac{\tan x + \tan 45^\circ}{1 - \tan x \tan 45^\circ}$$

$$= \frac{t+1}{1-t} = LHS$$

∴ proven.

12d (iii) $\tan(x+45^\circ) = \frac{4}{5}$

$$0 \le x \le 360^\circ$$

$$45^\circ \le x + 45^\circ \le 405^\circ$$

rel acute $= 38.659..$

$= 39^\circ$ (nearest deg)

Quads 1 and 3

$$x + 45^\circ = 180 + 39, 360 + 39^\circ$$

$$x = 174^\circ, 354^\circ$$

Question 13

13a) (i) Domain of $\sin^{-1}(x) = [-1, 1]$

so for $3\sin^{-1}\left(\frac{x}{2}\right)$

$$-1 \le \frac{x}{2} \le 1$$

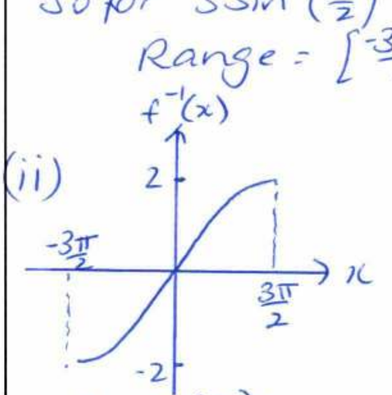
Domain $= [-2, 2]$

Range of $\sin^{-1}(x) = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

so for $3\sin^{-1}\left(\frac{x}{2}\right)$

Range $= \left[\frac{-3\pi}{2}, \frac{3\pi}{2}\right]$

(ii)



$y = 3\sin^{-1}\left(\frac{x}{2}\right)$

$f^{-1}(x) \rightarrow x = 3\sin^{-1}\left(\frac{y}{2}\right)$

$$\frac{y}{2} = \sin\left(\frac{x}{3}\right)$$

$$f^{-1}(x) = 2\sin\left(\frac{x}{3}\right)$$

13b) ①, 4, 7, 10, 13, ... , 94, 97, 100

104

104

Arith series $a = 1$ $d = 3$ $T_n = 100$

$$a + (n-1)d = 100$$

$$1 + 3n - 3 = 100$$

$$n = 34$$

∴ 34 values

16 pairs add to 104

+ number 1 + an extra in the middle.

∴ Could choose 16 digits with no pair + 2 digits (1 and extra) = 18 total

∴ Min is 19 by PHP.

13c) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

① Prove for $n = 1$

$LHS = \frac{1}{2!} = \frac{1}{2}$

$RHS = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2}$

∴ true for $n = 1$

② Assume for $n = k$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

③ Prove true for $n = k+1$

RTP $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$$= 1 - \frac{1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!}$$

$LHS = \left(\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k}{(k+1)!}\right) + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ by assumption

$$= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!}$$

→

13c) Cont'd

$$LHS = \frac{(k+2)! - k + 2 + k + 1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!}$$

$= RHS$ ∴ True for $n = k+1$

∴ Proven by MI

13d) $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{i}$

$\dot{\underset{\sim}{r}} = 2y\sqrt{y}\,\underset{\sim}{i} + y(1-y)\,\underset{\sim}{j}$

(i) $\frac{dy}{dt} = y(1-y)$

$$\int \frac{dy}{y(1-y)} = \int dt$$

$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right) dy = t + c$$

$$\ln|y| - \ln|1-y| = t + c$$

$$\ln\left|\frac{y}{1-y}\right| = t + c$$

$$e^{t+c} = \left|\frac{y}{1-y}\right|$$

Let $A = \pm e^c$

$Ae^t = \frac{y}{1-y}$ at $t=0, y=0.5$

$$A = \frac{0.5}{1-0.5} = 1$$

$$e^t = \frac{y}{1-y}$$

$$y = e^t - ye^t$$

$$y(1+e^t) = e^t$$

$$y = \frac{e^t}{1+e^t}$$

13d (ii) $|\dot{\underset{\sim}{r}}| = \sqrt{(2y\sqrt{y})^2 + y^2(1-y)^2}$

$$= \sqrt{4y^3 + y^2 - 2y^3 + y^4}$$

$$= \sqrt{y^4 + 2y^3 + y^2}$$

$$= y\sqrt{y^2 + 2y + 1}$$

$$= y\sqrt{(y+1)^2}$$

$$= y(y+1)$$

at $t = \ln 9$ $y = \frac{9}{1+9} = 0.9$

$\therefore |\dot{r}(0.9)| = 0.9(1.9) = 1.71$

Question 14

a)(i) $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{\sqrt{(-2)^2+3^2}} \times \begin{pmatrix}-2\\3\end{pmatrix}$

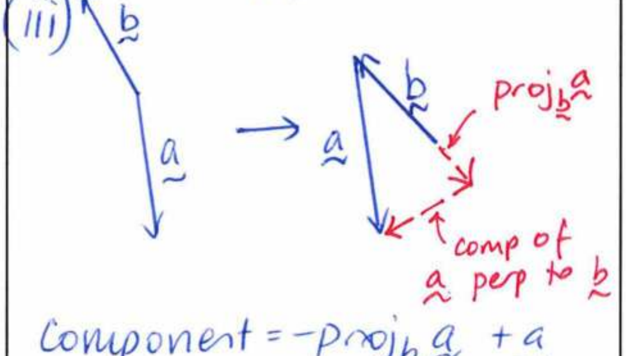
$$= \begin{pmatrix}\frac{-2}{\sqrt{13}}\\ \frac{3}{\sqrt{13}}\end{pmatrix}$$

(ii) $\text{Proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}||\underset{\sim}{b}|}\underset{\sim}{b}$

$$= \frac{-4-15}{\sqrt{13}\times\sqrt{13}}\begin{pmatrix}-2\\3\end{pmatrix}$$

$$= \begin{pmatrix}\frac{38}{13}\\ -\frac{57}{13}\end{pmatrix}$$

(iii)



$\text{Component} = -\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} + \underset{\sim}{a}$

$$= -\begin{pmatrix}\frac{38}{13}\\ \frac{-57}{13}\end{pmatrix} + \begin{pmatrix}2\\-5\end{pmatrix}$$

$$= \begin{pmatrix}\frac{-12}{13}\\ \frac{-8}{13}\end{pmatrix}$$

14b (i) $\frac{y}{2} = \ln x \rightarrow x = e^{\frac{y}{2}}$

$x^2 = e^y$

$V = \pi \int_0^h e^y dy$

$= \pi \left[e^y \right]_0^h$

$= \pi (e^h - 1)$ cm^3

(ii) $\frac{dh}{dt} = ?$ $\frac{dV}{dt} = 40$ $\frac{dV}{dh} = \pi e^h$ (from above)

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\pi e^h} \times 40 = \frac{40}{\pi e^h}$

at $t = 3$ $V = 120$

$120 = \pi(e^h - 1)$

$e^h = \frac{120}{\pi} + 1$

$\therefore \frac{dh}{dt} = \frac{40}{\pi\left(\frac{120}{\pi} + 1\right)}$

$= \frac{40}{120 + \pi}$ cm/min

14c (i) $x = 25t\cos\theta$

$\therefore t = \frac{x}{25\cos\theta}$ sub into y

$y = -4.9\left(\frac{x^2}{25^2\cos^2\theta}\right) + 25\left(\frac{x}{25\sin\theta}\right)\sin\theta$

$= \frac{-4.9}{625} x^2 \sec^2\theta + x\tan\theta$

$= -0.00784x^2(1 + \tan^2\theta) + x\tan\theta$

$= -0.00784x^2 - 0.00784\tan^2\theta + x\tan\theta$

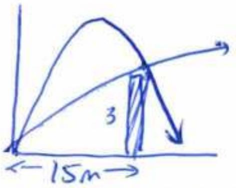
at $x = 15$, $y = 3$

$1.76\tan^2\theta - 15\tan\theta + 4.764 = 0$

$\tan\theta = \frac{15 \pm \sqrt{225 - 4 \times 4.764 \times 1.764}}{2 \times 1.764}$

$\theta = 83°1'28"$ or $18°17'8'$

14c (ii)



$\dot{x} = 25\cos\theta$

$\dot{y} = -9.8t + 25\sin\theta$

at $x = 15$, $\theta = 83°1'28"$

$t = \frac{15}{25\cos(83°1'28")}$

$\dot{x} = 25\cos(83°1'28")$

$\dot{y} = -9.8\left(\frac{3}{5\cos\theta}\right) + 25\sin\theta$

$= -23.60167812$

speed $= |\underset{\sim}{v}| = \sqrt{\dot{x}_1^2 + \dot{y}_1^2}$

$= 23.79616351$ ms^{-1}

$= 28.8$ ms^{-1} (1dp)

at $x = 15$ $\theta = 18°17'8"$

$\dot{x}_2 = 25\cos(18°17'8")$

$\dot{y}_2 = -9.8\left(\frac{3}{5\cos\theta}\right) + 25\sin\theta$

$= 1.651124345$

$|\underset{\sim}{v}| = \sqrt{\dot{x}_2^2 + \dot{y}_2^2}$

$= 23.794$

$= 23.8$ ms^{-1} (1dp)

$\therefore$ Both speeds are equal at that point.